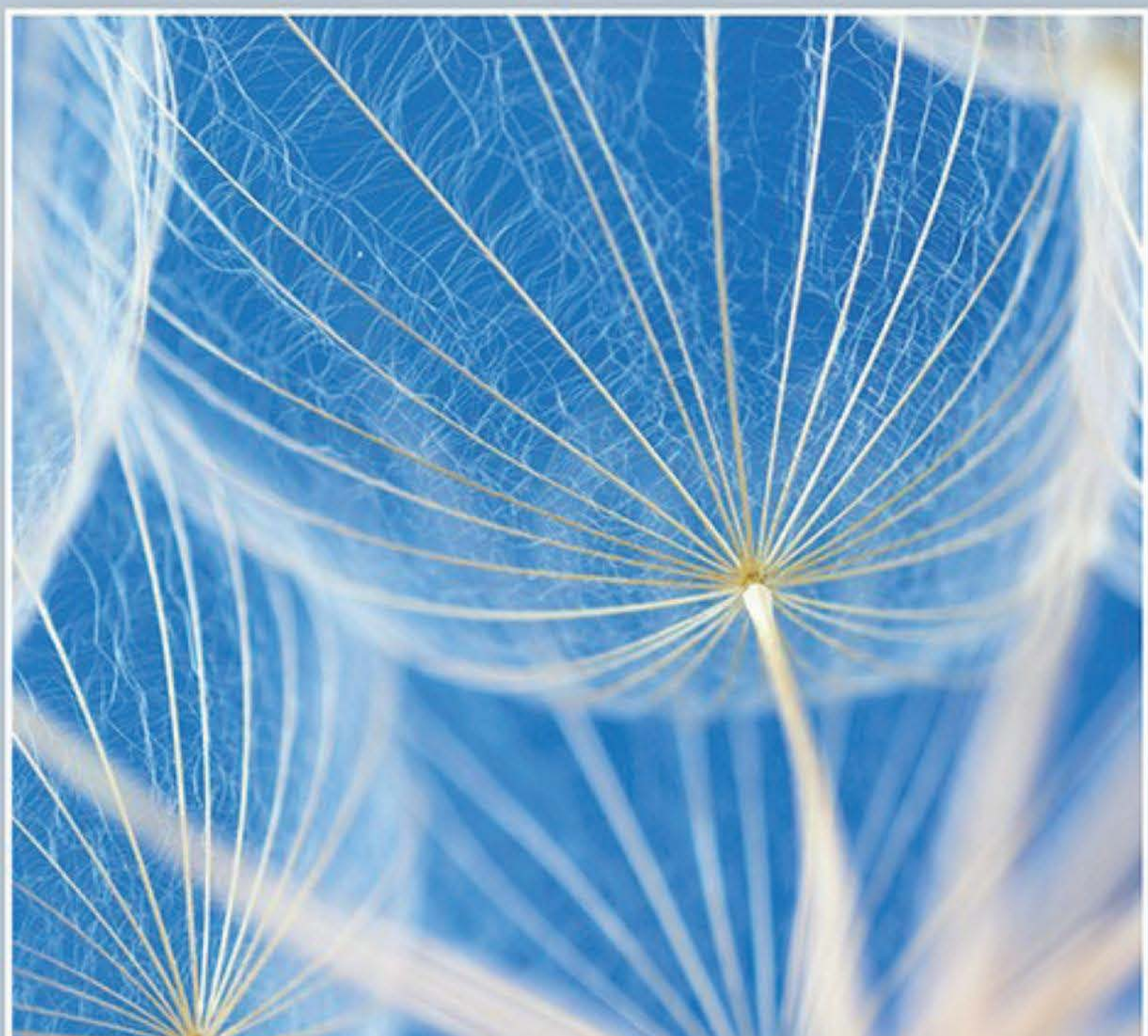
Probability and Statistical Inference

TENTH EDITION



Hogg | Tanis | Zimmerman

Discrete Distributions

Bernoulli
$0 < p < 1$

$$f(x) = p^x(1-p)^{1-x}, \qquad x = 0, 1$$
$$M(t) = 1 - p + pe^t, \qquad -\infty < t < \infty$$
$$\mu = p, \qquad \sigma^2 = p(1-p)$$

Binomial
$b(n, p)$
$n = 1, 2, 3, \ldots$
$0 < p < 1$

$$f(x) = \frac{n!}{x!(n-x)!} p^x(1-p)^{n-x}, \qquad x = 0, 1, 2, \ldots, n$$
$$M(t) = (1 - p + pe^t)^n, \qquad -\infty < t < \infty$$
$$\mu = np, \qquad \sigma^2 = np(1-p)$$

Geometric
$0 < p < 1$

$$f(x) = (1-p)^{x-1}p, \qquad x = 1, 2, 3, \ldots$$
$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < -\ln(1-p)$$
$$\mu = \frac{1}{p}, \qquad \sigma^2 = \frac{1-p}{p^2}$$

Hypergeometric
$N_1 > 0, \quad N_2 > 0$
$N = N_1 + N_2$
$1 \le n \le N_1 + N_2$

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \qquad x \le n,\ x \le N_1,\ n - x \le N_2$$
$$\mu = n\left(\frac{N_1}{N}\right), \qquad \sigma^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Negative Binomial
$0 < p < 1$
$r = 1, 2, 3, \ldots$

$$f(x) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, \qquad x = r, r+1, r+2, \ldots$$
$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \qquad t < -\ln(1-p)$$
$$\mu = r\left(\frac{1}{p}\right), \qquad \sigma^2 = \frac{r(1-p)}{p^2}$$

Poisson
$\lambda > 0$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots$$
$$M(t) = e^{\lambda(e^t - 1)}, \qquad -\infty < t < \infty$$
$$\mu = \lambda, \qquad \sigma^2 = \lambda$$

Uniform
$m = 1, 2, 3, \ldots$

$$f(x) = \frac{1}{m}, \qquad x = 1, 2, \ldots, m$$
$$\mu = \frac{m+1}{2}, \qquad \sigma^2 = \frac{m^2-1}{12}$$

Continuous Distributions

Beta $\alpha > 0$ $\beta > 0$	$f(x) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1$ $\mu = \dfrac{\alpha}{\alpha+\beta}, \qquad \sigma^2 = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$
Chi-square $\chi^2(r)$ $r = 1, 2, \ldots$	$f(x) = \dfrac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1}e^{-x/2}, \qquad 0 < x < \infty$ $M(t) = \dfrac{1}{(1-2t)^{r/2}}, \qquad t < \dfrac{1}{2}$ $\mu = r, \qquad \sigma^2 = 2r$
Exponential $\theta > 0$	$f(x) = \dfrac{1}{\theta} e^{-x/\theta}, \qquad 0 < x < \infty$ $M(t) = \dfrac{1}{1-\theta t}, \qquad t < \dfrac{1}{\theta}$ $\mu = \theta, \qquad \sigma^2 = \theta^2$
Gamma $\alpha > 0$ $\theta > 0$	$f(x) = \dfrac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1}e^{-x/\theta}, \qquad 0 < x < \infty$ $M(t) = \dfrac{1}{(1-\theta t)^\alpha}, \qquad t < \dfrac{1}{\theta}$ $\mu = \alpha\theta, \qquad \sigma^2 = \alpha\theta^2$
Normal $N(\mu, \sigma^2)$ $-\infty < \mu < \infty$ $\sigma > 0$	$f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$ $M(t) = e^{\mu t + \sigma^2 t^2/2}, \qquad -\infty < t < \infty$ $E(X) = \mu, \qquad \mathrm{Var}(X) = \sigma^2$
Uniform $U(a, b)$ $-\infty < a < b < \infty$	$f(x) = \dfrac{1}{b-a}, \qquad a \le x \le b$ $M(t) = \dfrac{e^{tb} - e^{ta}}{t(b-a)}, \qquad t \neq 0; \qquad M(0) = 1$ $\mu = \dfrac{a+b}{2}, \qquad \sigma^2 = \dfrac{(b-a)^2}{12}$

Probability and Statistical Inference

Tenth Edition

Robert V. Hogg

Elliot A. Tanis

Dale L. Zimmerman

Pearson

Director Portfolio Manager: Deirdre Lynch
Senior Portfolio Manager: Patrick Barbara
Portfolio Management Assistant: Morgan Danna
Content Producer: Tamela Ambush
Managing Producer: Karen Wernholm
Senior Producer: Aimee Thorne
Product Marketing Manager: Yvonne Vannatta
Field Marketing Manager: Evan St. Cyr
Marketing Assistant: Jon Bryant
Senior Author Support/Technology Specialist: Joe Vetere
Manager, Rights and Permissions: Gina Cheselka
Manufacturing Buyer: Carol Melville, LSC Communications
Cover Design, Composition, and Full Service Vendor: Pearson CSC
Full Service Project Management: Prince John - Pearson CSC
Cover Image: Alexandra Jursova/Moment/Getty Images

Library of Congress Cataloging-in-Publication Data

Names: Hogg, Robert V., author. | Tanis, Elliot A., author. | Zimmerman, Dale L., author.
Title: Probability and statistical inference / Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman.
Description: Tenth edition. | New York, NY : Pearson, 2018. | Includes bibliographical references and index.
Identifiers: LCCN 2018016546 | ISBN 9780135189399 | ISBN 013518939X
Subjects: LCSH: Probabilities. | Mathematical statistics.
Classification: LCC QA273 .H694 2018 | DDC 519.5–dc23 LC record available at https://urldefense.proofpoint.com/v2/url? u=https-3A__lccn.loc.gov_2018016546&d=DwIFAg&c=0YLnzTkWOdJlub_y7qAx8Q&r=wPGJMKK17CeHqRrBmHL3dANcU7vAP6xMwR_eHf4JPBM&m=grcVj3gROiayls_YZE61JriBkuhmiKnZjntz0t8EFQ4&s=OPgbnDS6cllWkFLqGuCc-CdBCX8__PFmMdvdQp5zyZw&e=

1 18

Pearson www.pearsonhighered.com

ISBN-13: 978-0-13-518939-9
ISBN-10: 0-13-518939-X

CONTENTS

PREFACE

In this Tenth Edition of *Probability and Statistical Inference*, Elliot Tanis and Dale Zimmerman would like to acknowledge the many contributions that Robert Hogg made to the first nine editions. Dr. Hogg died on December 23, 2014, but his insights continue on in this tenth edition. We are indebted to his influence on our lives and work.

CONTENT AND COURSE PLANNING

This text is designed for a two-semester course, but it can be adapted for a one-semester course. A good calculus background is needed, but no previous study of probability or statistics is required.

This new edition has more than 25 new examples and more than 75 new exercises. However, its chapters are organized in much the same manner as in the ninth edition. The first five again focus on probability, including the following topics: conditional probability, independence, Bayes' theorem, discrete and continuous distributions, certain mathematical expectations including moment-generating functions, bivariate distributions along with marginal and conditional distributions, correlation, functions of random variables and their distributions, the central limit theorem, and Chebyshev's inequality. We added a section on the hypergeometric distribution, adding to material that had previously been scattered throughout the first and second chapters. Also, to this portion of the book we added material on new topics, including the index of skewness and the laws of total probability for expectations and the variance. While the strong probability coverage of the first five chapters is important for all students, feedback we have received indicates that it has been particularly helpful to actuarial students who are studying for Exam P in the Society of Actuaries' series (or Exam 1 of the Casualty Actuarial Society).

The remaining four chapters of the book focus on statistical inference. Topics carried over from the previous edition include descriptive and order statistics, point estimation including maximum likelihood and method of moments estimation, sufficient statistics, Bayesian estimation, simple linear regression, interval estimation, and hypothesis testing. New material has been added on the topics of percentile matching and the invariance of maximum likelihood estimation, and we've added a new section on hypothesis testing for variances, which also includes confidence intervals for a variance and for the ratio of two variances. We present confidence intervals for means, variances, proportions, and regression coefficients; distribution-free confidence intervals for percentiles; and resampling methods (in particular, bootstrapping). Our coverage of hypothesis testing includes standard tests on means (including distribution-free tests), variances, proportions, and regression coefficients, power and sample size, best critical regions (Neyman-Pearson), and likelihood ratio tests. On the more applied side, we describe chi-square tests for goodness of fit and for association in contingency tables, analysis of variance including general factorial designs, and statistical quality control.

The first semester of the course should contain most of the topics in Chapters 1–5. The second semester includes some topics omitted there and many of those in Chapters 6–9. A more basic course might omit some of the starred sections, but we believe that the order of topics will give the instructor the flexibility needed in his or her course. The usual nonparametric and Bayesian techniques are placed at appropriate places in the text rather than in separate chapters. We find that many persons like the applications associated with statistical quality control in the last section.

The Prologue suggests many fields in which statistical methods can be used. At the end of each chapter, we give some interesting historical comments, which have proven to be very worthwhile in the past editions. The answers given in this text for exercises that involve the standard distributions were often calculated using our probability tables which, of course, are rounded off for printing. If you use a statistical package, your answers may differ slightly from those given.

ANCILLARIES

Data sets for this text are available on Pearson's Student Resources website: https://www.pearson.com/math-stats-resources.

An **Instructor's Solutions Manual** containing worked-out solutions to the even-numbered exercises in the text is available for download from Pearson Education Instructor's Resource website: https://www.pearson.com/us/sign-in.html.

Some of the numerical exercises were solved with *Maple*. For additional exercises that involve simulations, a separate manual, *Probability & Statistics: Explorations with MAPLE*, second edition, by Zaven Karian and Elliot Tanis, is available for download from Pearson's Student Resources website. This is located at https://www.pearson.com/math-stats-resources. Several exercises in that manual also make use of the power of *Maple* as a computer algebra system.

If you find errors in this text, please send them to dale-zimmerman@uiowa.edu so that they can be corrected in a future printing. These **errata** will also be posted on http://homepage.divms.uiowa.edu/~dzimmer/.

ACKNOWLEDGMENTS

We wish to thank our colleagues, students, and friends for many suggestions and for their generosity in supplying data for exercises and examples. In particular we would like thank the reviewers of the ninth edition who made suggestions for this edition. They are Maureen Cox from St. Bonaventure University, Lynne Seymour from the University of Georgia, Kevin Keen from the University of Northern British Columbia, Clifford Taylor from Concordia University Ann Arbor, Melanie Autin from Western Kentucky University, Aubie Anisef from Douglas College, Manohar Aggarwal from the University of Memphis, Joseph Huber from the University of Kansas, and Christopher Swanson from Ashland University. Mark Mills from Central College in Iowa, Matthew Bognar from the University of Iowa, and David Schweitzer from Liberty University also made many helpful comments, and Hongda Zhang of the University of Iowa wrote solutions to some of the new exercises. We also acknowledge the excellent suggestions from our copy editor Jody Callahan and the fine work of our accuracy checker Kyle Siegrist. We also thank the University of Iowa and Hope College for providing office space and encouragement. Finally, our families have been most understanding during the preparation of all of this material.

We would especially like to thank our wives, Elaine and Bridget. We truly appreciate their patience and needed their love.

Elliot A. Tanis
tanis@hope.edu

Dale L. Zimmerman
dale-zimmerman@uiowa.edu

This page intentionally left blank

PROLOGUE

The discipline of statistics deals with the collection and analysis of data. Advances in computing technology, particularly in relation to changes in science and business, have increased the need for more statistical scientists to examine the huge amount of data being collected. We know that data are not equivalent to information. Once data (hopefully of high quality) are collected, there is a strong need for statisticians to make sense of them. That is, data must be analyzed in order to provide information upon which decisions can be made. In light of this great demand, opportunities for the discipline of statistics have never been greater, and there is a special need for more bright young persons to go into statistical science.

If we think of fields in which data play a major part, the list is almost endless: accounting, actuarial science, atmospheric science, biological science, economics, educational measurement, environmental science, epidemiology, finance, genetics, manufacturing, marketing, medicine, pharmaceutical industries, psychology, sociology, sports, and on and on. Because statistics is useful in all these areas, it really should be taught as an applied science. Nevertheless, to go very far in such an applied science, it is necessary to understand the importance of creating models for each situation under study. Now, no model is ever exactly right, but some are extremely useful as an approximation to the real situation. To be applied properly, most appropriate models in statistics require a certain mathematical background in probability. Accordingly, while alluding to applications in the examples and exercises, this textbook is really about the mathematics needed for the appreciation of probabilistic models necessary for statistical inferences.

In a sense, statistical techniques are really the heart of the scientific method. Observations are made that suggest conjectures. These conjectures are tested, and data are collected and analyzed, providing information about the truth of the conjectures. Sometimes the conjectures are supported by the data, but often the conjectures need to be modified and more data must be collected to test the modifications, and so on. Clearly, in this iterative process, statistics plays a major role with its emphasis on proper design and analysis of experiments and the resulting inferences upon which decisions can be made. Through statistics, information is provided that is relevant to taking certain actions, including improving manufactured products, providing better services, marketing new products or services, forecasting energy needs, classifying diseases better, and so on.

Statisticians recognize that there are often errors in their inferences, and they attempt to quantify the probabilities of those mistakes and make them as small as possible. That these uncertainties even exist is due to the fact that there is variation in the data. Even though experiments are repeated under seemingly the same conditions, the results vary from trial to trial. In light of this uncertainty, the statistician tries to summarize the data in the best possible way, always explaining the error structures of the statistical estimates.

This is an important lesson to be learned: Variation is almost everywhere. It is the statistician's job to understand variation. Often, as in manufacturing, the desire is to reduce variation so that the products will be more consistent. In other words, car doors will fit better in the manufacturing of automobiles if the variation is decreased by making each door closer to its target values.

Any student of statistics should understand the nature of variability and the necessity for creating probabilistic models of that variability. We cannot avoid making inferences and decisions in the face of this uncertainty; however, these inferences and decisions are greatly influenced by the probabilistic models selected. Some persons are better model builders than others and accordingly will make better inferences and decisions. The assumptions needed for each statistical model are carefully examined; it is hoped that thereby the reader will become a better model builder.

Finally, we must mention how modern statistical analyses have become dependent upon the computer. Increasingly, statisticians and computer scientists are working together in areas of exploratory data analysis and "data mining." Statistical software development is critical today, for the best of it is needed in complicated data analyses. In light of this growing relationship between these two fields, it is good advice for bright students to take substantial offerings in statistics and in computer science.

Students majoring in statistics, computer science, or a program at their interface such as data science are in great demand in the workplace and in graduate programs. Clearly, they can earn advanced degrees in statistics or computer science or both. But, more important, they are highly desirable candidates for graduate work in other areas: actuarial science, industrial engineering, finance, marketing, accounting, management science, psychology, economics, law, sociology, medicine, health sciences, etc. So many fields have been "mathematized" that their programs are begging for majors in statistics or computer science. Often, such students become "stars" in these other areas. We truly hope that we can interest students enough that they want to study more statistics. If they do, they will find that the opportunities for very successful careers are numerous.

Chapter

1 PROBABILITY

1.1 PROPERTIES OF PROBABILITY

It is usually difficult to explain to the general public what statisticians do. Many think of us as "math nerds" who seem to enjoy dealing with numbers, and there is some truth to that concept. But if we consider the bigger picture, many recognize that statisticians can be extremely helpful in many investigations.

Consider the following:

1. There is some problem or situation that needs to be considered, so statisticians are often asked to work with investigators or research scientists.
2. Suppose that some measure is needed to help us understand the situation better. The measurement problem is often extremely difficult, and creating good measures is a valuable skill. As an illustration, in higher education, how do we measure good teaching? This is a question to which we have not found a satisfactory answer, although several measures such as student evaluations have been used in the past.
3. After the measuring instrument has been developed, we must collect data through observation; this data could possibly be the results of a survey or experiment.
4. Using these data, statisticians summarize the results, often with descriptive statistics and graphical methods.
5. These summaries are then used to analyze the situation. Here, it is possible that statisticians can make what are called statistical inferences.
6. Finally, a report is presented, along with some recommendations that are based upon the data and the analysis of them. Frequently, such a recommendation might be to perform the survey or experiment again, possibly changing some of the questions or factors involved. This is how statistics is used in what is referred to as the scientific method, because often the analysis of the data suggests other experiments. Accordingly, the scientist must consider different possibilities in his or her search for an answer and thus perform similar experiments over and over again.

The discipline of statistics deals with the *collection* and *analysis of data*. When measurements are taken, even seemingly under the same conditions, the results usually vary. Despite this variability, a statistician tries to find a pattern. However, due to the "noise," not all the data fit into the pattern. In the face of the variability, the statistician must still determine the best way to describe the pattern. Accordingly, statisticians know that mistakes will be made in data analysis, and they try to minimize those errors as much as possible and then give bounds on the possible errors. By considering these bounds, decision-makers can decide how much confidence they want to place in the data and in their analysis. If the bounds are wide, perhaps more data should be collected. If, however, the bounds are narrow, the person involved in the study might want to make a decision and proceed accordingly.

Variability is a fact of life, and proper statistical methods can help us understand data collected under inherent variability. Because of this variability, many decisions have to be made that involve uncertainties. For example, a medical researcher's interest may center on the effectiveness of a new vaccine for mumps; an agronomist must decide whether an increase in yield can be attributed to a new strain of wheat; a meteorologist is interested in predicting the probability of rain; the state legislature must decide whether decreasing speed limits will result in fewer accidents; the admissions officer of a college must predict the college performance of an incoming freshman; a biologist is interested in estimating the clutch size for a particular type of bird; an economist desires to estimate the unemployment rate; and an environmentalist tests whether new controls have resulted in a reduction in pollution.

In reviewing the preceding (relatively short) list of possible areas of statistical applications, the reader should recognize that good statistics is closely associated with careful thinking in many investigations. As an illustration, students should appreciate how statistics is used in the endless cycle of the scientific method. We observe nature and ask questions, we run experiments and collect data that shed light on these questions, we analyze the data and compare the results of the analysis with what we previously thought, we raise new questions, and on and on. Or, if you like, statistics is clearly part of the important "plan–do–study–act" cycle: Questions are raised and investigations planned and carried out. The resulting data are studied, analyzed, and then acted upon, often raising new questions.

There are many aspects of statistics. Some people get interested in the subject by collecting data and trying to make sense of their observations. In some cases the answers are obvious and little training in statistical methods is necessary. But if a person goes very far in many investigations, he or she soon realizes that there is a need for some theory to help describe the error structure associated with the various estimates of the patterns. That is, at some point appropriate probability and mathematical models are required to make sense of complicated datasets. Statistics and the probabilistic foundation on which statistical methods are based can provide the models to help people do this. So, in this book we are more concerned with the mathematical rather than the applied aspects of statistics. Still, we give enough real examples so that the reader can get a good sense of a number of important applications of statistical methods.

In the study of statistics, we consider experiments for which the outcome cannot be predicted with certainty. Such experiments are called **random experiments**. Although the specific outcome of a random experiment cannot be predicted with certainty before the experiment is performed, the collection of all possible outcomes *is* known and can be described and perhaps listed. The collection of all possible outcomes is denoted by S and is called the **sample space**. Given a sample space S, let A be a part of the collection of outcomes in S; that is, $A \subset S$. Then A is called an **event**.

When the random experiment is performed and the outcome of the experiment is in A, we say that **event** A **has occurred**.

In studying probability, the words *set* and *event* are interchangeable, so the reader might want to review **algebra of sets**. Here we remind the reader of some terminology:

- $\emptyset$ denotes the **null** or **empty** set;
- $A \subset B$ means A is a **subset** of B;
- $A \cup B$ is the **union** of A and B;
- $A \cap B$ is the **intersection** of A and B;
- A' is the **complement** of A (i.e., all elements in S that are not in A).

Some of these sets are depicted by the shaded regions in Figure 1.1-1, in which S is the interior of the rectangles. Such figures are called **Venn diagrams**.

Special terminology associated with events that is often used by statisticians includes the following:

1. $A_1, A_2, \ldots, A_k$ are **mutually exclusive events**, meaning that $A_i \cap A_j = \emptyset$, $i \neq j$; that is, $A_1, A_2, \ldots, A_k$ are disjoint sets;
2. $A_1, A_2, \ldots, A_k$ are **exhaustive events**, meaning that $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

So, if $A_1, A_2, \ldots, A_k$ are **mutually exclusive and exhaustive** events, we know that $A_i \cap A_j = \emptyset$, $i \neq j$, and $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Set operations satisfy several properties. For example, if A, B, and C are subsets of S, we have the following:

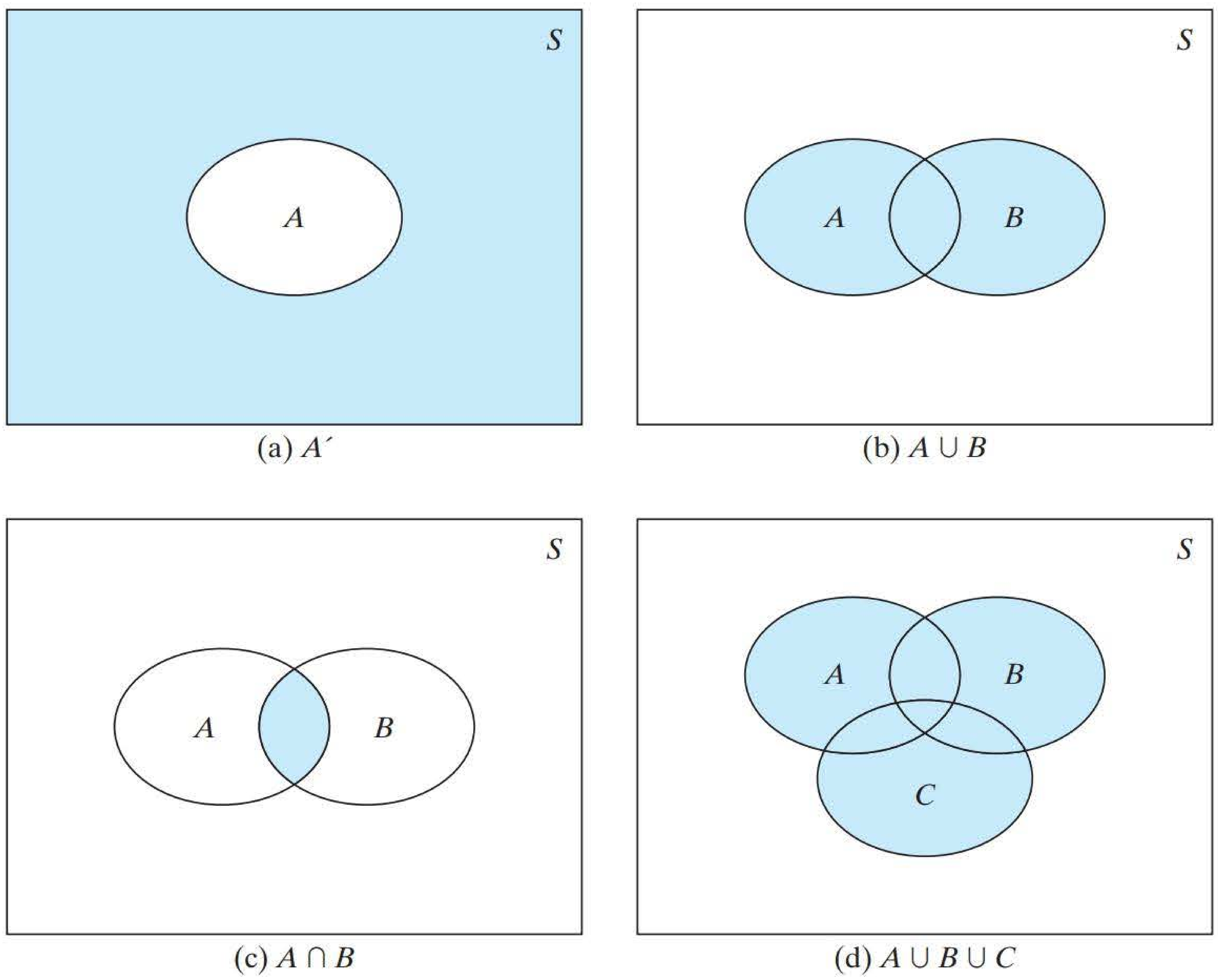


Figure 1.1-1 Algebra of sets

Commutative Laws

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associative Laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$
$$(A \cap B)' = A' \cup B'$$

A Venn diagram will be used to justify the first of De Morgan's laws. In Figure 1.1-2(a), $A \cup B$ is represented by horizontal lines, and thus $(A \cup B)'$ is the region represented by vertical lines. In Figure 1.1-2(b), A' is indicated with horizontal lines, and B' is indicated with vertical lines. An element belongs to $A' \cap B'$ if it belongs to both A' and B'. Thus the crosshatched region represents $A' \cap B'$. Clearly, this crosshatched region is the same as that shaded with vertical lines in Figure 1.1-2(a).

We are interested in defining what is meant by the probability of event A, denoted by $P(A)$ and often called the chance of A occurring. To help us understand what is meant by the probability of A, consider repeating the experiment a number of times—say, n times. We call these repetitions **trials**. Count the number of times that event A actually occurred throughout these n trials; this number is called the frequency of event A and is denoted by $\mathcal{N}(A)$. The ratio $\mathcal{N}(A)/n$ is called the **relative frequency** of event A in these n repetitions of the experiment. A relative frequency is usually very unstable for small values of n, but it tends to stabilize as n increases. This suggests that we associate with event A a number—say, p—that is equal to the number about which the relative frequency tends to stabilize. This number p can then be taken as the number that the relative frequency of event A will be near in future

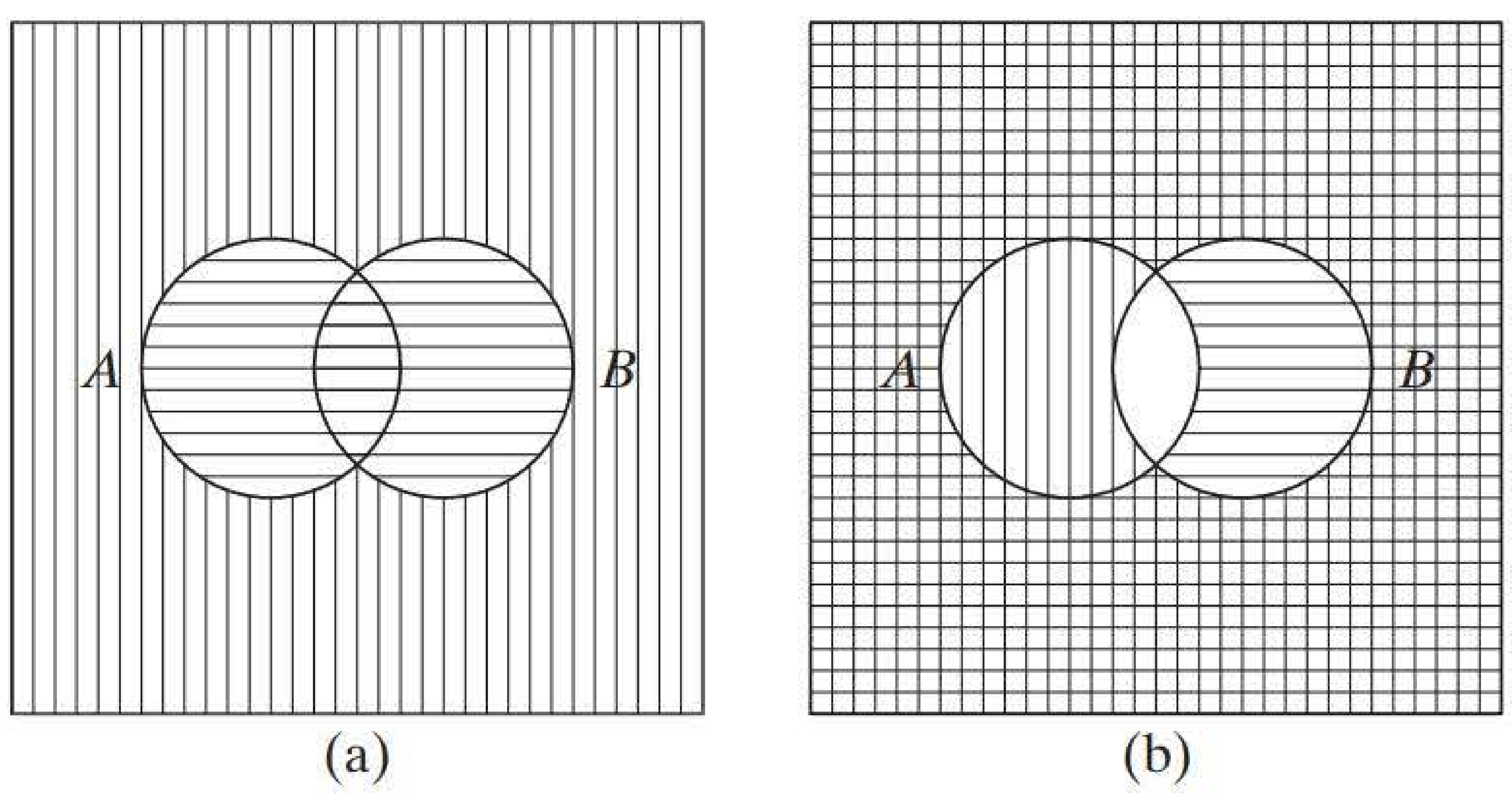


Figure 1.1-2 Venn diagrams illustrating De Morgan's laws

performances of the experiment. Thus, although we cannot predict the outcome of a random experiment with certainty, if we know p, then for a large value of n we can predict fairly accurately the relative frequency associated with event A. The number p assigned to event A is called the **probability** of event A and is denoted by $P(A)$. That is, $P(A)$ represents the proportion of outcomes of a random experiment that terminate in the event A as the number of trials of that experiment increases without bound.

The next example will help to illustrate some of the ideas just presented.

Example 1.1-1 A fair six-sided die is rolled six times. If the face numbered k is the outcome on roll k for $k = 1, 2, \ldots, 6$, we say that a match has occurred. The experiment is called a success if at least one match occurs during the six trials. Otherwise, the experiment is called a failure. The sample space is $S = \{\text{success, failure}\}$. Let $A = \{\text{success}\}$. We would like to assign a value to $P(A)$. Accordingly, this experiment was simulated 500 times on a computer. Figure 1.1-3 depicts the results of this simulation, and the following table summarizes a few of the results:

n	$\mathcal{N}(A)$	$\mathcal{N}(A)/n$
50	37	0.740
100	69	0.690
250	172	0.688
500	330	0.660

The probability of event A is not intuitively obvious, but it will be shown in Example 1.4-6 that $P(A) = 1 - (1 - 1/6)^6 = 0.665$. This assignment is certainly supported by the simulation (although not proved by it). ■

Example 1.1-1 shows that, at times, intuition cannot be used to assign probabilities, although simulation can perhaps help to assign a probability empirically. The next example illustrates where intuition can help in assigning a probability to an event.

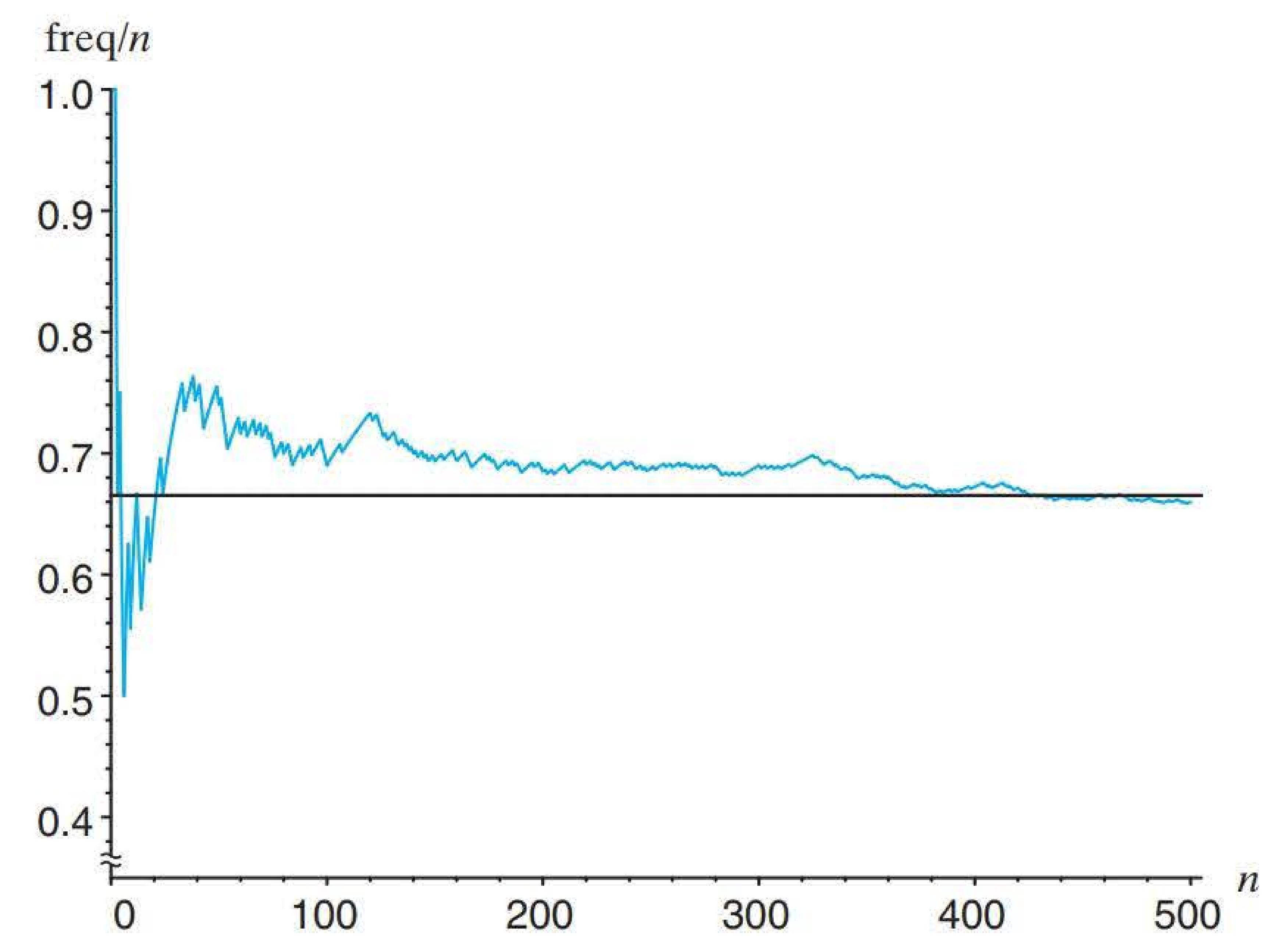


Figure 1.1-3 Fraction of experiments having at least one match

Example 1.1-2 A disk 2 inches in diameter is thrown at random on a tiled floor, where each tile is a square with sides 4 inches in length. Let C be the event that the disk will land entirely on one tile. In order to assign a value to $P(C)$, consider the center of the disk. In what region must the center lie to ensure that the disk lies entirely on one tile? If you draw a picture, it should be clear that the center must lie within a square having sides of length 2 and with its center coincident with the center of a tile. Because the area of this square is 4 and the area of a tile is 16, it makes sense to let $P(C) = 4/16$. ■

Sometimes the nature of an experiment is such that the probability of A can be assigned easily. For example, when a state lottery randomly selects a three-digit integer, we would expect each of the 1000 possible three-digit numbers to have the same chance of being selected, namely, 1/1000. If we let $A = \{233, 323, 332\}$, then it makes sense to let $P(A) = 3/1000$. Or if we let $B = \{234, 243, 324, 342, 423, 432\}$, then we would let $P(B) = 6/1000$. Obtaining probabilities of events associated with many other random experiments is not as straightforward as this; Example 1.1-1 provided such a case.

So we wish to associate with A a number $P(A)$ about which the relative frequency $\mathcal{N}(A)/n$ of the event A tends to stabilize with large n. A function such as $P(A)$ that is evaluated for a set A is called a **set function**. In this section, we consider the probability set function $P(A)$ and discuss some of its properties. In succeeding sections, we will describe how the probability set function is defined for particular experiments.

To help decide what properties the probability set function should satisfy, consider properties possessed by the relative frequency $\mathcal{N}(A)/n$. For example, $\mathcal{N}(A)/n$ is always nonnegative. If $A = S$, the sample space, then the outcome of the experiment will always belong to S, and thus $\mathcal{N}(S)/n = 1$. Also, if A and B are two mutually exclusive events, then $\mathcal{N}(A \cup B)/n = \mathcal{N}(A)/n + \mathcal{N}(B)/n$. Hopefully, these remarks will help to motivate the following definition.

Definition 1.1-1

Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

(a) $P(A) \geq 0$;

(b) $P(S) = 1$;

(c) if $A_1, A_2, A_3, \ldots$ are events and $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

for each positive integer k, and

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

for an infinite, but countable, number of events.

The theorems that follow give some other important properties of the probability set function. When one considers these theorems, it is important to understand the theoretical concepts and proofs. However, if the reader keeps the relative frequency concept in mind, the theorems should also have some intuitive appeal.

Theorem 1.1-1 For each event A,

$$P(A) = 1 - P(A').$$

Proof [See Figure 1.1-1(a).] We have

$$S = A \cup A' \qquad \text{and} \qquad A \cap A' = \emptyset.$$

Thus, from properties (b) and (c), it follows that

$$1 = P(A) + P(A').$$

Hence

$$P(A) = 1 - P(A'). \qquad \square$$

Example 1.1-3 A fair coin is flipped successively until the same face is observed on successive flips. Let $A = \{x : x = 3, 4, 5, \ldots\}$; that is, A is the event that it will take three or more flips of the coin to observe the same face on two consecutive flips. Perhaps the easiest way to find $P(A)$ is to first find the probability of $A' = \{x : x = 2\}$, the complement of A. In two flips of a coin, the possible outcomes are $\{HH, HT, TH, TT\}$, and because the coin is fair it is reasonable to assume that each of these four outcomes has the same chance of being observed. Thus,

$$P(A') = P(\{HH, TT\}) = \frac{2}{4}.$$

It follows from Theorem 1.1-1 that

$$P(A) = 1 - P(A') = 1 - \frac{2}{4} = \frac{2}{4}. \qquad \blacksquare$$

Theorem 1.1-2 $P(\emptyset) = 0$.

Proof In Theorem 1.1-1, take $A = \emptyset$ so that $A' = S$. Then

$$P(\emptyset) = 1 - P(S) = 1 - 1 = 0. \qquad \square$$

Theorem 1.1-3 If events A and B are such that $A \subset B$, then $P(A) \le P(B)$.

Proof Because $A \subset B$, we have

$$B = A \cup (B \cap A') \qquad \text{and} \qquad A \cap (B \cap A') = \emptyset.$$

Hence, from property (c),

$$P(B) = P(A) + P(B \cap A') \ge P(A)$$

because, from property (a),

$$P(B \cap A') \ge 0. \qquad \square$$

Theorem 1.1-4 For each event A, $P(A) \leq 1$.

Proof Because $A \subset S$, we have, by Theorem 1.1-3 and property (b),

$$P(A) \leq P(S) = 1,$$

which gives the desired result. □

Property (a), along with Theorem 1.1-4, shows that, for each event A,

$$0 \leq P(A) \leq 1.$$

Theorem 1.1-5 If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof [See Figure 1.1-1(b).] The event $A \cup B$ can be represented as a union of mutually exclusive events, namely,

$$A \cup B = A \cup (A' \cap B).$$

Hence, by property (c),

$$P(A \cup B) = P(A) + P(A' \cap B). \tag{1.1-1}$$

However,

$$B = (A \cap B) \cup (A' \cap B),$$

which is a union of mutually exclusive events. Thus,

$$P(B) = P(A \cap B) + P(A' \cap B)$$

and

$$P(A' \cap B) = P(B) - P(A \cap B).$$

If the right side of this equation is substituted into Equation 1.1-1, we obtain

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

which is the desired result. □

Example 1.1-4 Among a certain population of men, 30% are smokers, 40% are obese, and 25% are both smokers and obese. Suppose we select a man at random from this population, and we let A be the event that he is a smoker and B be the event that he is obese. Then

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.30 + 0.40 - 0.25 \ = \ 0.45 \end{aligned}$$

is the probability that the selected man is either a smoker or obese. ■

Theorem 1.1-6 If A, B, and C are any three events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ -P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Proof [See Figure 1.1-1(d).] Write

$$A \cup B \cup C = A \cup (B \cup C)$$

and apply Theorem 1.1-5. The details are left as an exercise. □

Example 1.1-5 A survey was taken of a group's viewing habits of sporting events on TV during the last year. Let $A = \{\text{watched football}\}$, $B = \{\text{watched basketball}\}$, and $C = \{\text{watched baseball}\}$. The results indicate that if a person is selected at random from the surveyed group, then $P(A) = 0.43$, $P(B) = 0.40$, $P(C) = 0.32$, $P(A \cap B) = 0.29$, $P(A \cap C) = 0.22$, $P(B \cap C) = 0.20$, and $P(A \cap B \cap C) = 0.15$. It then follows that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\quad -P(B \cap C) + P(A \cap B \cap C) \\ &= 0.43 + 0.40 + 0.32 - 0.29 - 0.22 - 0.20 + 0.15 \\ &= 0.59 \end{aligned}$$

is the probability that this person watched at least one of these sports. ■

Let a probability set function be defined on a sample space $S = \{e_1, e_2, \ldots, e_m\}$, where each e_i is a possible outcome of the experiment. The integer m is called the total number of ways in which the random experiment can terminate. If each of these outcomes has the same probability of occurring, we say that the m outcomes are **equally likely**. That is,

$$P(\{e_i\}) = \frac{1}{m}, \qquad i = 1, 2, \ldots, m.$$

If the number of outcomes in an event A is h, then the integer h is called the number of ways that are favorable to the event A. In this case, $P(A)$ is equal to the number of ways favorable to the event A divided by the total number of ways in which the experiment can terminate. That is, under this assumption of equally likely outcomes, we have

$$P(A) = \frac{h}{m} = \frac{N(A)}{N(S)},$$

where $h = N(A)$ is the number of ways A can occur and $m = N(S)$ is the number of ways S can occur. Exercise 1.1-15 considers this assignment of probability in a more theoretical manner.

It should be emphasized that in order to assign the probability h/m to the event A, we must assume that each of the outcomes $e_1, e_2, \ldots, e_m$ has the same probability $1/m$. This assumption is then an important part of our probability model; if it is not realistic in an application, then the probability of the event A cannot be computed in this way. Actually, we have used this result in the simple case given in Example 1.1-3 because it seemed realistic to assume that each of the possible outcomes in $S = \{HH, HT, TH, TT\}$ had the same chance of being observed.

Example 1.1-6 Let a card be drawn at random from an ordinary deck of 52 playing cards. Then the sample space S is the set of $m = 52$ different cards, and it is reasonable to assume that each of these cards has the same probability of selection, $1/52$. Accordingly, if A is the set of outcomes that are kings, then $P(A) = 4/52 = 1/13$ because there are $h = 4$ kings in the deck. That is, $1/13$ is the probability of drawing a card that is a king, provided that each of the 52 cards has the same probability of being drawn. ■

In Example 1.1-6, the computations are very easy because there is no difficulty in determining the appropriate values of h and m. However, instead of drawing only one card, suppose that 13 are taken at random and without replacement. Then we can think of each possible 13-card hand as being an outcome in a sample space, and it is reasonable to assume that each of these outcomes has the same probability. For example, using the preceding method to assign the probability of a hand consisting of seven spades and six hearts, we must be able to count the number h of all such hands as well as the number m of possible 13-card hands. In these more complicated situations, we need better methods of determining h and m. We discuss some of these counting techniques in Section 1.2.

Exercises

1.1-1. Of a group of patients having injuries, 28% visit both a physical therapist and a chiropractor while 8% visit neither. Say that the probability of visiting a physical therapist exceeds the probability of visiting a chiropractor by 16%. What is the probability of a randomly selected person from this group visiting a physical therapist?

1.1-2. An insurance company looks at its auto insurance customers and finds that (a) all insure at least one car, (b) 85% insure more than one car, (c) 23% insure a sports car, and (d) 17% insure more than one car, including a sports car. Find the probability that a customer selected at random insures exactly one car and it is not a sports car.

1.1-3. Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns $1/52$ to each of the 52 outcomes. Let

$$A = \{x\colon x \text{ is a jack, queen, or king}\},$$

$$B = \{x\colon x \text{ is a 9, 10, or jack and } x \text{ is red}\},$$

$$C = \{x\colon x \text{ is a club}\},$$

$$D = \{x\colon x \text{ is a diamond, a heart, or a spade}\}.$$

Find **(a)** $P(A)$, **(b)** $P(A \cap B)$, **(c)** $P(A \cup B)$, **(d)** $P(C \cup D)$, and **(e)** $P(C \cap D)$.

1.1-4. A fair coin is tossed four times, and the sequence of heads and tails is observed.

(a) List each of the 16 sequences in the sample space S.

(b) Let events A, B, C, and D be given by $A = \{\text{at least 3 heads}\}$, $B = \{\text{at most 2 heads}\}$, $C = \{\text{heads on the third toss}\}$, and $D = \{\text{1 head and 3 tails}\}$. If the probability set function assigns $1/16$ to each outcome in the sample space, find **(i)** $P(A)$, **(ii)** $P(A \cap B)$, **(iii)** $P(B)$, **(iv)** $P(A \cap C)$, **(v)** $P(D)$, **(vi)** $P(A \cup C)$, and **(vii)** $P(B \cap D)$.

1.1-5. Consider the trial on which a 3 is first observed in successive rolls of a six-sided die. Let A be the event that 3 is observed on the first trial. Let B be the event that at least two trials are required to observe a 3. Assuming that each side has probability $1/6$, find **(a)** $P(A)$, **(b)** $P(B)$, and **(c)** $P(A \cup B)$.

1.1-6. If $P(A) = 0.5$, $P(B) = 0.6$, and $P(A \cap B) = 0.4$, find **(a)** $P(A \cup B)$, **(b)** $P(A \cap B')$, and **(c)** $P(A' \cup B')$.

1.1-7. Given that $P(A \cup B) = 0.76$ and $P(A \cup B') = 0.87$, find $P(A)$.

1.1-8. During a visit to a primary care physician's office, the probability of having neither lab work nor referral to a specialist is 0.21. Of those coming to that office, the probability of having lab work is 0.41 and the probability of having a referral is 0.53. What is the probability of having both lab work and a referral?

1.1-9. Roll a fair six-sided die three times. Let $A_1 = \{\text{1 or 2 on the first roll}\}$, $A_2 = \{\text{3 or 4 on the second roll}\}$, and $A_3 = \{\text{5 or 6 on the third roll}\}$. It is given that $P(A_i) = 1/3$, $i = 1, 2, 3$; $P(A_i \cap A_j) = (1/3)^2$, $i \neq j$; and $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

(a) Use Theorem 1.1-6 to find $P(A_1 \cup A_2 \cup A_3)$.

(b) Show that $P(A_1 \cup A_2 \cup A_3) = 1 - (1 - 1/3)^3$.

1.1-10. Prove Theorem 1.1-6.

1.1-11. A typical roulette wheel used in a casino has 38 slots that are numbered $1, 2, 3, \ldots, 36, 0, 00$, respectively. The 0 and 00 slots are colored green. Half of the remaining slots are red and half are black. Also, half of the integers between 1 and 36 inclusive are odd, half are even, and 0 and 00 are defined to be neither odd nor even. A ball is rolled around the wheel and ends up in one of the slots; we assume that each slot has equal probability of 1/38, and we are interested in the number of the slot into which the ball falls.

(a) Define the sample space S.

(b) Let $A = \{0, 00\}$. Give the value of $P(A)$.

(c) Let $B = \{14, 15, 17, 18\}$. Give the value of $P(B)$.

(d) Let $D = \{x : x \text{ is odd}\}$. Give the value of $P(D)$.

1.1-12. Let x equal a number that is selected randomly from the closed interval from zero to one, $[0, 1]$. Use your intuition to assign values to

(a) $P(\{x\colon 0 \le x \le 1/3\})$.

(b) $P(\{x\colon 1/3 \le x \le 1\})$.

(c) $P(\{x\colon x = 1/3\})$.

(d) $P(\{x\colon 1/2 < x < 5\})$.

1.1-13. Divide a line segment into two parts by selecting a point at random. Use your intuition to assign a probability to the event that the longer segment is at least two times longer than the shorter segment.

1.1-14. Let the interval $[-r, r]$ be the base of a semicircle. If a point is selected at random from this interval, assign a probability to the event that the length of the perpendicular segment from the point to the semicircle is less than $r/2$.

1.1-15. Let $S = A_1 \cup A_2 \cup \cdots \cup A_m$, where events $A_1, A_2, \ldots, A_m$ are mutually exclusive and exhaustive.

(a) If $P(A_1) = P(A_2) = \cdots = P(A_m)$, show that $P(A_i) = 1/m,\ i = 1, 2, \ldots, m$.

(b) If $A = A_1 \cup A_2 \cup \cdots \cup A_h$, where $h < m$, and (a) holds, prove that $P(A) = h/m$.

1.1-16. Let $p_n,\ n = 0, 1, 2, \ldots$, be the probability that an automobile policyholder will file for n claims in a five-year period. The actuary involved makes the assumption that $p_{n+1} = (1/4)p_n$. What is the probability that the holder will file two or more claims during this period?

1.2 METHODS OF ENUMERATION

In this section, we develop counting techniques that are useful in determining the number of outcomes associated with the events of certain random experiments. We begin with a consideration of the multiplication principle.

Multiplication Principle: Suppose that an experiment (or procedure) E_1 has n_1 outcomes and, for each of these possible outcomes, an experiment (procedure) E_2 has n_2 possible outcomes. Then the composite experiment (procedure) E_1E_2 that consists of performing first E_1 and then E_2 has n_1n_2 possible outcomes.

Example 1.2-1 Let E_1 denote the selection of a rat from a cage containing one female (F) rat and one male (M) rat. Let E_2 denote the administering of either drug A (A), drug B (B), or a placebo (P) to the selected rat. Then the outcome for the composite experiment can be denoted by an ordered pair, such as (F, P). In fact, the set of all possible outcomes, namely, $(2)(3) = 6$ of them, can be denoted by the following rectangular array:

$$\begin{array}{ccc} (\text{F, A}) & (\text{F, B}) & (\text{F, P}) \\ (\text{M, A}) & (\text{M, B}) & (\text{M, P}) \end{array}$$

Another way of illustrating the multiplication principle is with a tree diagram such as that in Figure 1.2-1. The diagram shows that there are $n_1 = 2$ possibilities (branches) for the sex of the rat and that, for each of these outcomes, there are $n_2 = 3$ possibilities (branches) for the drug.

Clearly, the multiplication principle can be extended to a sequence of more than two experiments or procedures. Suppose that the experiment E_i has

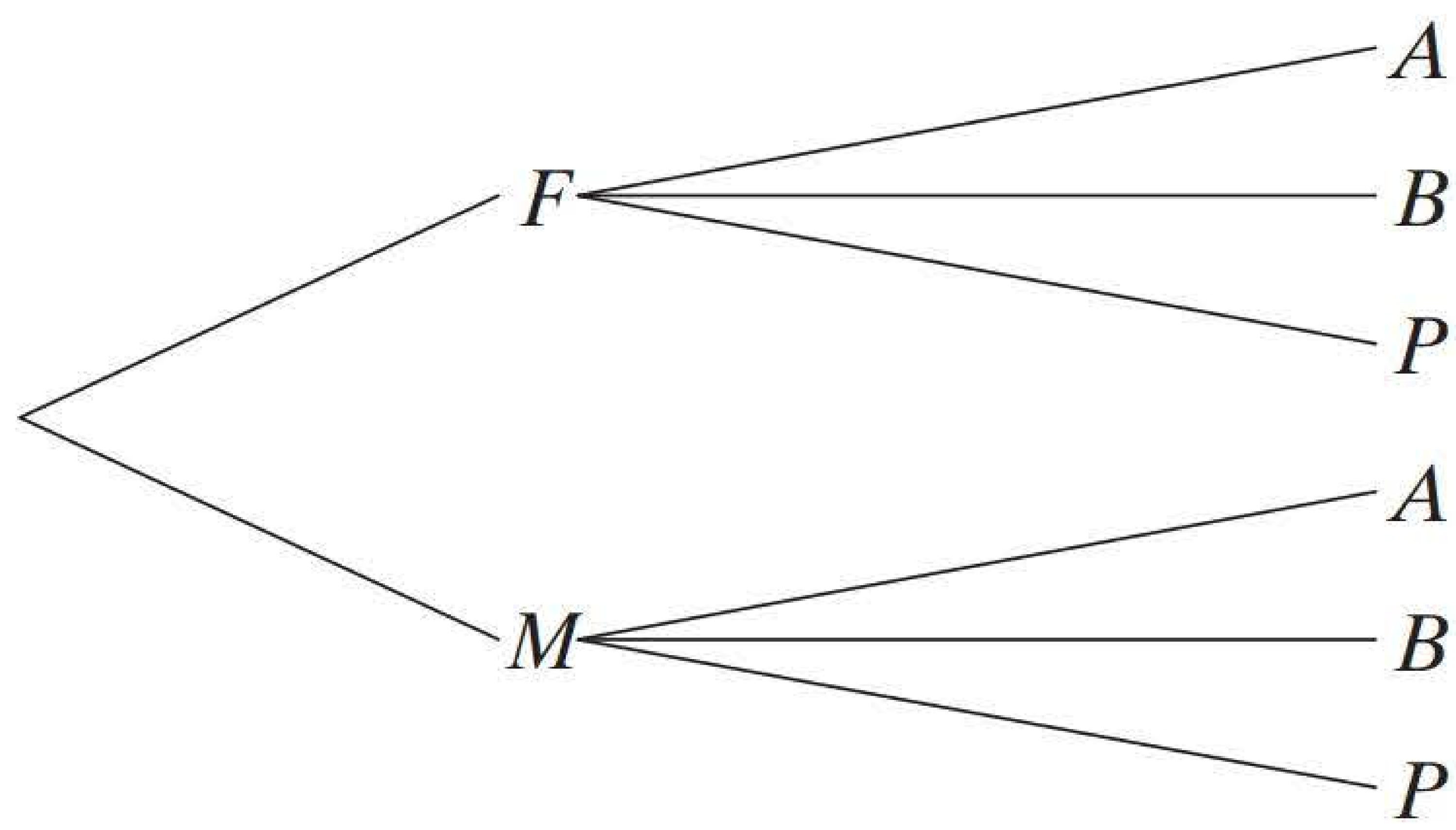


Figure 1.2-1 Tree diagram

n_i ($i = 1, 2, \ldots, m$) possible outcomes after previous experiments have been performed. Then the composite experiment $E_1E_2 \cdots E_m$ that consists of performing E_1, then E_2, ..., and finally E_m has $n_1n_2 \cdots n_m$ possible outcomes.

Example 1.2-2 A cafe lets you order a deli sandwich your way. There are: E_1, six choices for bread; E_2, four choices for meat; E_3, four choices for cheese; and E_4, 12 different garnishes (condiments). The number of different sandwich possibilities, if you may choose one bread, 0 or 1 meat, 0 or 1 cheese, and from 0 to 12 condiments is, by the multiplication principle, noting that you may choose or not choose each condiment,

$$(6)(5)(5)(2^{12}) = 614{,}400$$

different sandwich combinations.

Although the multiplication principle is fairly simple and easy to understand, it will be extremely useful as we now develop various counting techniques.

Suppose that n positions are to be filled with n different objects. There are n choices for filling the first position, $n-1$ for the second, ..., and 1 choice for the last position. So, by the multiplication principle, there are

$$n(n-1)\cdots(2)(1) = n!$$

possible arrangements. The symbol $n!$ is read "n factorial." We define $0! = 1$; that is, we say that zero positions can be filled with zero objects in one way.

Definition 1.2-1
Each of the $n!$ arrangements (in a row) of n different objects is called a **permutation** of the n objects.

Example 1.2-3 The number of permutations of the four letters a, b, c, and d is clearly $4! = 24$. However, the number of possible four-letter code words using the four letters a, b, c, and d if letters may be repeated is $4^4 = 256$, because in this case each selection can be performed in four ways.

If only r positions are to be filled with objects selected from n different objects, $r \le n$, then the number of possible ordered arrangements is

$${}_nP_r = n(n-1)(n-2)\cdots(n-r+1).$$

That is, there are n ways to fill the first position, $(n-1)$ ways to fill the second, and so on, until there are $[n-(r-1)] = (n-r+1)$ ways to fill the rth position.

In terms of factorials, we have

$$ {}_nP_r = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots(3)(2)(1)}{(n-r)\cdots(3)(2)(1)} = \frac{n!}{(n-r)!}. $$

Definition 1.2-2
Each of the ${}_nP_r$ arrangements is called a **permutation of n objects taken r at a time**.

Example 1.2-4 The number of possible four-letter code words, selecting from the 26 letters in the alphabet, in which all four letters are different is

$$ {}_{26}P_4 = (26)(25)(24)(23) = \frac{26!}{22!} = 358{,}800. $$

Example 1.2-5 The number of ways of selecting a president, a vice president, a secretary, and a treasurer in a club consisting of ten persons is

$$ {}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!} = 5040. $$

Suppose that a set contains n objects. Consider the problem of selecting r objects from this set. The order in which the objects are selected may or may not be important. In addition, it is possible that a selected object is replaced before the next object is selected. Accordingly, we give some definitions and show how the multiplication principle can be used to count the number of possibilities.

Definition 1.2-3
If r objects are selected from a set of n objects, and if the order of selection is noted, then the selected set of r objects is called an **ordered sample of size r**.

Definition 1.2-4
Sampling with replacement occurs when an object is selected and then replaced before the next object is selected.

By the multiplication principle, the number of possible ordered samples of size r taken from a set of n objects is n^r when sampling with replacement.

Example 1.2-6 A die is rolled seven times. The number of possible ordered samples is $6^7 = 279{,}936$. Note that rolling a die is equivalent to sampling with replacement from the set $\{1,2,3,4,5,6\}$.

Example 1.2-7 An urn contains ten balls numbered 0, 1, 2, ..., 9. If four balls are selected one at a time and with replacement, then the number of possible ordered samples is $10^4 = 10{,}000$. Note that this is the number of four-digit integers between 0000 and 9999, inclusive. ■

Definition 1.2-5
Sampling without replacement occurs when an object is not replaced after it has been selected.

By the multiplication principle, the number of possible ordered samples of size r taken from a set of n objects without replacement is

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!},$$

which is equivalent to ${}_nP_r$, the number of permutations of n objects taken r at a time.

Example 1.2-8 The number of ordered samples of five cards that can be drawn without replacement from a standard deck of 52 playing cards is

$$(52)(51)(50)(49)(48) = \frac{52!}{47!} = 311{,}875{,}200.$$ ■

REMARK Note that it must be true that $r \le n$ when sampling without replacement, but r can exceed n when sampling with replacement. ■

Often the order of selection is not important and interest centers only on the selected set of r objects. That is, we are interested in the number of subsets of size r that can be selected from a set of n different objects. In order to find the number of (unordered) subsets of size r, we count, in two different ways, the number of ordered subsets of size r that can be taken from the n distinguishable objects. Then, equating the two answers, we are able to count the number of (unordered) subsets of size r.

Let C denote the number of (unordered) subsets of size r that can be selected from n different objects. We can obtain each of the ${}_nP_r$ ordered subsets by first selecting one of the C unordered subsets of r objects and then ordering these r objects. Because the latter ordering can be carried out in $r!$ ways, the multiplication principle yields $(C)(r!)$ ordered subsets; so $(C)(r!)$ must equal ${}_nP_r$. Thus, we have

$$(C)(r!) = \frac{n!}{(n-r)!},$$

or

$$C = \frac{n!}{r!\,(n-r)!}.$$

We denote this answer by either ${}_nC_r$ or $\binom{n}{r}$; that is,

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Accordingly, a set of n different objects possesses

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

unordered subsets of size $r \le n$.

We could also say that the number of ways in which r objects can be selected without replacement from n objects when the order of selection is disregarded is $\binom{n}{r} = {}_nC_r$, and the latter expression can be read as "n choose r." This result motivates the next definition.

Definition 1.2-6
Each of the ${}_nC_r$ unordered subsets is called a **combination of n objects taken r at a time**, where

$${}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Example 1.2-9 The number of possible five-card hands (in five-card poker) drawn from a deck of 52 playing cards is

$${}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960.$$

Example 1.2-10 The number of possible 13-card hands (in bridge) that can be selected from a deck of 52 playing cards is

$${}_{52}C_{13} = \binom{52}{13} = \frac{52!}{13!\,39!} = 635{,}013{,}559{,}600.$$

The numbers $\binom{n}{r}$ are frequently called **binomial coefficients** because they arise in the expansion of a binomial. We illustrate this property by giving a justification of the binomial expansion

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} b^r a^{n-r}. \tag{1.2-1}$$

For each summand in the expansion of

$$(a+b)^n = (a+b)(a+b)\cdots(a+b),$$

either an a or a b is selected from each of the n factors. One possible product is then $b^r a^{n-r}$; this occurs when b is selected from each of r factors and a from each of the remaining $n-r$ factors. But the latter operation can be completed in $\binom{n}{r}$ ways, which then must be the coefficient of $b^r a^{n-r}$, as shown in Equation 1.2-1.

The binomial coefficients are given in Table I in Appendix B for selected values of n and r. Note that for some combinations of n and r, the table uses the fact that

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = \binom{n}{n-r}.$$

That is, the number of ways in which r objects can be selected out of n objects is equal to the number of ways in which $n-r$ objects can be selected out of n objects.

Example 1.2-11

Assume that each of the $\binom{52}{5} = 2{,}598{,}960$ five-card hands drawn from a deck of 52 playing cards has the same probability of being selected. Then the number of possible five-card hands that are all spades (event A) is

$$N(A) = \binom{13}{5}\binom{39}{0},$$

because the five spades can be selected from the 13 spades in $\binom{13}{5}$ ways, after which zero nonspades can be selected in $\binom{39}{0} = 1$ way. We have

$$\binom{13}{5} = \frac{13!}{5!8!} = 1287$$

from Table I in Appendix B. Thus, the probability of an all-spade five-card hand is

$$P(A) = \frac{N(A)}{N(S)} = \frac{1287}{2{,}598{,}960} = 0.000495.$$

Suppose now that the event B is the set of outcomes in which exactly three cards are kings and exactly two cards are queens. We can select the three kings in any one of $\binom{4}{3}$ ways and the two queens in any one of $\binom{4}{2}$ ways. By the multiplication principle, the number of outcomes in B is

$$N(B) = \binom{4}{3}\binom{4}{2}\binom{44}{0},$$

where $\binom{44}{0}$ gives the number of ways in which 0 cards are selected out of the nonkings and nonqueens and of course is equal to 1. Thus,

$$P(B) = \frac{N(B)}{N(S)} = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{0}}{\binom{52}{5}} = \frac{24}{2{,}598{,}960} = 0.0000092.$$

Finally, let C be the set of outcomes in which there are exactly two kings, two queens, and one jack. Then

$$P(C) = \frac{N(C)}{N(S)} = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{40}{0}}{\binom{52}{5}} = \frac{144}{2{,}598{,}960} = 0.000055$$

because the numerator of this fraction is the number of outcomes in C. ■

Now suppose that a set contains n objects of two types: r of one type and $n - r$ of the other type. The number of permutations of n different objects is $n!$. However, in this case, the objects are not all distinguishable. To count the number of distinguishable arrangements, first select r out of the n positions for the objects of the first type. This can be done in $\binom{n}{r}$ ways. Then fill in the remaining positions with the objects of the second type. Thus, the number of distinguishable arrangements is

$$_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Definition 1.2-7
Each of the $_nC_r$ permutations of n objects, r of one type and $n - r$ of another type, is called a **distinguishable permutation**.

Example 1.2-12 A coin is flipped ten times and the sequence of heads and tails is observed. The number of possible 10-tuplets that result in four heads and six tails is

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10!}{6!\,4!} = \binom{10}{6} = 210.$$

■

Example 1.2-13 The Eastern red-backed salamander (*Plethodon cinereus*) is the most abundant salamander species in the northeastern forests of North America and has three recognized color morphs: red-backed, lead-backed, and erythristic (red with black mottling). In a study of possible differences among color morphs with respect to their ability to avoid detection and capture by predators, four red-backed and five lead-backed morphs are released into a controlled forest environment. The order in which they are captured (and eaten) by predators is recorded. Considering only the color morph of the salamanders, there are

$$\binom{9}{4} = \frac{9!}{5!4!} = 126$$

possible orders in which they can be captured. ■

The foregoing results can be extended. Suppose that in a set of n objects, n_1 are similar, n_2 are similar, ..., n_s are similar, where $n_1 + n_2 + \cdots + n_s = n$. Then the number of distinguishable permutations of the n objects is (see Exercise 1.2-15)

$$\binom{n}{n_1, n_2, \ldots, n_s} = \frac{n!}{n_1!\,n_2!\,\cdots\,n_s!}. \tag{1.2-2}$$

Example 1.2-14 Among eight Eastern red-backed salamanders released into a controlled forest environment, three are the red-backed morphs, three are lead-backed, and two are erythristic. The number of possible orders in which these salamanders can be captured by predators is

$$\binom{8}{3, 3, 2} = \frac{8!}{3!\,3!\,2!} = 560.$$

■